Entanglement in the interaction between two quantum oscillator systems

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The fundamental quantum dynamics of two interacting oscillator systems are studied in two different scenarios. In one case, both oscillators are assumed to be linear, whereas in the second case, one oscillator is linear and the other is a non-linear, angular-momentum oscillator; the second case is, of course, more complex in terms of energy transfer and dynamics. These two scenarios have been the subject of much interest over the years, especially in developing an understanding of modern concepts in quantum optics and quantum electronics. In this work, however, these two scenarios are utilized to consider and discuss the salient features of quantum behaviors resulting from the interactive nature of the two oscillators, i.e., coherence, entanglement, spontaneous emission, etc., and to apply a *measure of entanglement* in analyzing the nature of the interacting systems.

The Heisenberg equation for both coupled oscillator scenarios are developed in terms of the relevant reduced kinematics operator variables and parameterized commutator relations. For the second scenario, by setting the relevant commutator relations to one or zero, respectively, the Heisenberg equations are able to describe the full quantum or classical motion of the interaction system, thus allowing us to discern the differences between the fully quantum and fully classical dynamical picture.

For the coupled linear and angular-momentum oscillator system in the fully quantum-mechanical description, we consider special examples of two, three, four-level angular momentum systems, demonstrating the explicit appearances of entanglement. We also show that this entanglement persists even as the *coupled* angular momentum oscillator is taken to the limit of a large number of levels, a limit which would go over to the classical picture for an *uncoupled* angular momentum oscillator.

Key words: entanglement: coupled-boson representation: spontaneous emission

1. INTRODUCTION

In this paper, we study the quantum dynamics of two interacting oscillator systems in two different, but very well-known scenarios. In case one, both oscillators are assumed to be linear, and in case two, one oscillator is assumed to be linear, while the other oscillator is assumed to be non-linear (angular momentum oscillator). We study Heisenberg equations and the time evolution of the wave-functions $|\psi(t)\rangle = e^{-i\,\hat{H}\,t/\hbar}\,|\psi(0)\rangle$ for both cases to discuss the quantum interactive behaviors between the two oscillators therein, e.g., entanglement, spontaneous emission (in case two), and quantum versus classical differences (in case two). Entanglement, an implicit non-classical ingredient of interacting quantum systems, is the key to enabling quantum computation, which allows for the solution of certain classes of problems more efficiently than the classical counterpart (see, e.g., Refs. 1,2). For case two, by setting the relevant commutator relation in each oscillator to one or zero, we utilize an interesting quantum-classical scheme[3] to clearly characterize the quantum features in the system.

In order to investigate the time evolution of the wave-function and the notion of entanglement in case one, we adopt the coupled-boson representation,[4] which was discussed by Schwinger in terms of the correspondence between two one-dimensional linear oscillators and an angular momentum oscillator. Based on this representation, we will easily obtain exact energy eigenvalues and eigenstates of case one. The coupled-boson representation has been used in context with similar types of problems to construct an ideal model for quantum-mechanical interference[5] and to study the current operator in two coupled linear oscillator (the rate of exchange of occupation number between oscillators).[6]

We adopt here $1-\text{Tr}\,(\hat{\rho}^{(\nu)})^2$ as an entanglement measure for the total system in a pure state,[7] where $\hat{\rho}^{(\nu)}$ is a reduced density matrix of individual oscillator ν. We will show that this measure increases with the occupation number n for a given initial state for case one, and with the maximum quantum number j of the angular momentum oscillator for case two, respectively; this increase with occupation number results from considering a dissipationless and coherent process, whereas we know that entanglement will diminish when loss is taken into account, e.g., resulting from the (thermal) interaction between the system and its environment.[8] For case two, we will discuss entanglement for the cases where the angular momentum oscillator is in the states $j = \frac{1}{2},\, 1,\, \frac{3}{2}$ explicitly.

In section 2, we study the interaction between two linear oscillators. The Heisenberg equations of motion are analyzed in the coupled boson representation, leading to the straightforward diagonalization of the coupled oscillator Hamiltonian; from this diagonalized Hamiltonian, the exact eigenstates and eigenvalues of the coupled oscillator

Hamiltonian are obtained, thus allowing for the calculation of the reduced density matrix, and the subsequent study of the entanglement under the two initial conditions of simple product state (a disentangled state) and exact eigenstate for the coupled oscillator system (an entangled state).

In section 3, we study the interaction between a linear and a non-linear, angular momentum oscillator. We make use of the seminal work of Ref. 3 to analyze the quantum-mechanical and the classical dynamics of this interaction, and to connect the quantum manifestation of spontaneous emission in this system with the notion of entanglement. Further, the total system Hamiltonian is diagonalized for $j = \frac{1}{2}, 1, \frac{3}{2}$ using rotational generating operators of the group $SU(n)$.[9] From this diagonalization, the reduced density matrix, and thus the measure of entanglement, is calculated for simple product (disentangled) initial states. Unlike the linear-linear oscillator system, this coupled system displays periodicity in the measure of entanglement for $j = \frac{1}{2}$ and $j = 1$, whereas for $j = \frac{3}{2}$ and beyond, the measure of entanglement is aperiodic; this aperiodicity is shown to be a result of the complex structure of the diagonalized multi-level Hamiltonian.

2. INTERACTION OF TWO LINEAR OSCILLATORS

2.1. Quantum behavior in the Heisenberg picture

Let us begin with case one. The coupled oscillator system under investigation is described in the *rotating wave approximation* by the Hamiltonian

$$\hat{H} \;=\; \hbar\omega_1 \left(\hat{a}_1^\dagger\, \hat{a}_1 \,+\, \tfrac{1}{2}\mathbb{1}\right) \,+\, \hbar\omega_2 \left(\hat{a}_2^\dagger\, \hat{a}_2 \,+\, \tfrac{1}{2}\mathbb{1}\right) \,+\, \hbar\,\kappa \left(\hat{a}_1^\dagger\, \hat{a}_2 \,+\, \hat{a}_1\, \hat{a}_2^\dagger\right)\,, \tag{1}$$

where κ denotes a coupling strength, and we assume that $\omega_1 \geq \omega_2$. Schwinger found that the quantum-mechanical angular momentum can be obtained by using creation and annihilation operators of two one-dimensional harmonic oscillators in terms of $\hat{\mathcal{J}}_l \propto \hat{a}^\dagger \otimes \hat{a}$. This is explicitly given as follows:[4]

$$\hat{\mathcal{J}}_x \;\equiv\; \tfrac{\hbar}{2}\left(\hat{a}_1^\dagger \otimes \hat{a}_2 + \hat{a}_1 \otimes \hat{a}_2^\dagger\right),\; \hat{\mathcal{J}}_y \;\equiv\; \tfrac{\hbar}{2i}\left(\hat{a}_1^\dagger \otimes \hat{a}_2 - \hat{a}_1 \otimes \hat{a}_2^\dagger\right),\; \hat{\mathcal{J}}_z \;\equiv\; \tfrac{\hbar}{2}\left(\hat{n}_1 - \hat{n}_2\right),\; \hat{\mathcal{J}} \;\equiv\; \tfrac{\hbar}{2}\left(\hat{n}_1 + \hat{n}_2\right)\,, \tag{2}$$

where $\hat{n}_\nu = \hat{a}_\nu^\dagger\, \hat{a}_\nu$ with $\nu = 1, 2$ is obviously an occupation number operator of each linear oscillator ν. From $\left[\hat{a}_\mu\,, \hat{a}_\nu^\dagger\right] = \mathbb{1}\,\delta_{\mu\nu}$ the operators $\hat{\mathcal{J}}_l$, where $l = x, y, z$, satisfy the usual angular-momentum commutation relations. The operator $\hat{\mathcal{J}}$ yields the quantum number of angular momentum, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$. In terms of the angular momentum operators in (2) the Hamiltonian in (1) is rewritten as

$$\hat{H} \;=\; \Delta\omega\, \hat{\mathcal{J}}_z \,+\, 2\,\kappa\, \hat{\mathcal{J}}_x \,+\, (\omega_1 + \omega_2)\; \hat{\mathcal{J}} \,+\, \tfrac{\mathbb{1}}{2}\,\hbar\,(\omega_1 + \omega_2)\,, \tag{3}$$

where $\Delta\omega \equiv \omega_1 - \omega_2 \geq 0$. Since $[\hat{\mathcal{J}}_l, \hat{\mathcal{J}}\,] = 0$, we easily obtain a constant of motion $\hat{\mathcal{J}}$ with $j = \frac{1}{2}(n_1 + n_2)$:

$$i\,\hbar\,\dot{\hat{\mathcal{J}}} \;=\; \left[\hat{\mathcal{J}}\,, \hat{H}\right] \;=\; 0\,, \tag{4}$$

which means that the total oscillator occupation number is conserved. Also, we find that $\dot{\hat{\mathcal{J}}}_x = -\Delta\omega\, \hat{\mathcal{J}}_y$ and so $\hat{\mathcal{J}}_x$ is a constant of motion for $\Delta\omega = 0$.

With the aid of the identity,

$$e^{\vartheta\hat{U}}\, \hat{V}\, e^{-\vartheta\hat{U}} \;=\; \hat{V} \,+\, \vartheta[\hat{U}, \hat{V}] \,+\, \tfrac{\vartheta^2}{2!}\,[\hat{U}, [\hat{U}, \hat{V}]] \,+\, \tfrac{\vartheta^3}{3!}\,[\hat{U}, [\hat{U}, [\hat{U}, \hat{V}]]] \,+\, \cdots\,, \tag{5}$$

we evaluate $e^{i\gamma\hat{\mathcal{J}}_y(0)/\hbar}\, \hat{H}\, e^{-i\gamma\hat{\mathcal{J}}_y(0)/\hbar}$, where $0 \leq \gamma \leq \frac{\pi}{2}$, to arrive at the diagonalized Hamiltonian of eq. (3) as

$$\hat{H}_d \;=\; e^{i\gamma\hat{\mathcal{J}}_y(0)/\hbar}\, \hat{H}\, e^{-i\gamma\hat{\mathcal{J}}_y(0)/\hbar} \;=\; \tfrac{\mathbb{1}}{2}\,\hbar\,(\omega_1 + \omega_2) \,+\, \{\Delta\omega \cdot (\cos\gamma) \,+\, 2\kappa \cdot (\sin\gamma)\}\; \hat{\mathcal{J}}_z(0) \,+\, (\omega_1 \,+\, \omega_2)\; \hat{\mathcal{J}}\,, \tag{6}$$

where $\tan\gamma \equiv \frac{2\kappa}{\Delta\omega}$ (note that in case of $\Delta\omega = 0$, we have $\gamma = \frac{\pi}{2}$ for any non-zero κ). From eqs. (2), (6) we now obtain two renormalized uncoupled harmonic oscillators with the renormalized energy splitting for each oscillator,

$$\hat{H}_d \;=\; \hbar\,\omega_1' \,\left(\hat{A}_1^\dagger\, \hat{A}_1 \,+\, \tfrac{\mathbb{1}}{2}\right) \,+\, \hbar\,\omega_2' \,\left(\hat{A}_2^\dagger\, \hat{A}_2 \,+\, \tfrac{\mathbb{1}}{2}\right) \tag{7}$$

where $\omega_1' \equiv \frac{1}{2}(\omega_1+\omega_2)+\frac{1}{2}\bar{\omega}$, $\omega_2' \equiv \frac{1}{2}(\omega_1+\omega_2)-\frac{1}{2}\bar{\omega}$. Here, $\bar{\omega}^2 = (2\kappa)^2+(\Delta\omega)^2$, and $\hat{A}_\nu(t) \equiv e^{-i\gamma\hat{\mathcal{J}}_y(0)/\hbar}\, \hat{a}_\nu(t)\, e^{i\gamma\hat{\mathcal{J}}_y(0)/\hbar}$ explicitly reads

$$\begin{pmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \end{pmatrix} \;=\; \hat{R}\left(\tfrac{-\gamma}{2}\right) \cdot \begin{pmatrix} \hat{a}_1(t) \\ \hat{a}_2(t) \end{pmatrix} \tag{8}$$

where the rotation matrix is

$$\hat{R}(\vartheta) \;=\; \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} . \tag{9}$$

Thus, from the Schwinger approach, we realize that the coupling term in eq. (1) can be equivalently represented as the $\hat{\mathcal{J}}_x$ component of angular momentum, and therefore, $\hat{H}$ in eq. (1) can be diagonalized by a unitary rotational operator with respect to the angular momentum operator $\hat{\mathcal{J}}_y(0)$ through γ.[6] Then, using eqs. (7), (8) we easily obtain the original annihilation operators as

$$\hat{a}_\nu(t) \;=\; \left\{\left(\cos\tfrac{\gamma}{2}\right)^2 e^{-i\omega_\nu' t} \,+\, \left(\sin\tfrac{\gamma}{2}\right)^2 e^{-i\omega_\mu' t}\right\} \hat{a}_\nu(0) \,+\, \tfrac{1}{2}\, (\sin\gamma)\, \left(e^{-i\omega_1' t} \,-\, e^{-i\omega_2' t}\right) \hat{a}_\mu(0)\,, \tag{10}$$

where $\nu, \mu = 1,2$ but $\nu \neq \mu$. Here, neglecting the "non-local" observable $\hat{a}_\mu(0)$, we obviously have $[\hat{a}_\nu(t), \hat{a}_\nu^\dagger(t)] \neq 1$ at $t \neq 0$. Now, we easily acquire the exact eigenstates and eigenvalues of the Hamiltonian in (1) as

$$|n_1, n_2\rangle_H \;\equiv\; e^{-i\gamma\hat{\mathcal{J}}_y(0)/\hbar}\, |n_1\rangle\, |n_2\rangle\,;\;\; E_{n_1,n_2} \;=\; \tfrac{\hbar}{2}\,(n_1+n_2+1)\,(\omega_1+\omega_2) \,+\, \tfrac{\hbar}{2}\,(n_1-n_2)\,\bar{\omega}\,. \tag{11}$$

We see here that for $\gamma = \frac{\pi}{2}$ (i.e., $\Delta\omega = 0$), the eigenstates $|n_1, n_2\rangle_H$ will also be eigenstates of $\hat{\mathcal{J}}_x(0) = \hat{\mathcal{J}}_x(t)$.

2.2. Appearance of the entanglement in the Schrödinger picture

Let us now study the entanglement in the time evolution of the two coupled linear oscillators, $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle$. To this end, we adopt,[7] as a measure of entanglement,

$$\mathcal{M}_{|\psi(t)\rangle} \;=\; 1 \,-\, \mathcal{P}\left[\hat{\rho}^{(\nu)}(t)\right] , \tag{12}$$

where $\mathcal{P}\left[\hat{\rho}^{(\nu)}\right] \;\equiv\; \mathrm{Tr}_\nu\,(\hat{\rho}^{(\nu)})^2$ is the purity measure of the state of each oscillator ν. Here, it clearly holds that $\mathcal{P}[\hat{\rho}^{(1)}] = \mathcal{P}[\hat{\rho}^{(2)}]$.[10] As can also easily be verified, $\mathcal{P}[\hat{\rho}^{(\nu)}]$ takes on a maximum value of 1 if the oscillator ν is in a pure state; it then follows that $\mathcal{M}_{|\psi\rangle} = 0$ for a simple product (disentangled) state, $|\psi\rangle = |\psi^{(1)}\rangle|\phi^{(2)}\rangle$. On the other hand, $\mathcal{P}[\hat{\rho}^{(\nu)}]$ takes on its minimum value $\frac{1}{M}$, where M is a number of accessible orthogonal states $\{|p_\nu\rangle; p = 1, 2, \cdots, M\}$ of each oscillator ν, if the oscillator ν is in the maximally mixed state given by $\hat{\rho}_{kl}^{(\nu)} = \frac{1}{M}\,\delta_{kl}$; then, the corresponding total wave-function $|\psi\rangle$ is a maximally entangled state in form of $\frac{1}{\sqrt{M}}\,(|1_1\rangle|1_2\rangle + |2_1\rangle|2_2\rangle + \cdots + |M_1\rangle|M_2\rangle)$, and $\mathcal{M}_{|\psi\rangle}$ has its maximum value $\frac{M-1}{M}$. We will explicitly determine $\mathcal{M}_{|\psi(t)\rangle}$ in its time evolution with mathematically manageable initial states $|\psi(0)\rangle$ given below.

We obtain the reduced density matrix, $\hat{\rho}^{(\nu)}(t)$, from the system state evolution, $|\psi(t)\rangle$, by constructing the system density matrix $\hat{\rho}(t) = |\psi(t)\rangle\,\langle\psi(t)|$ from the known state of the system, and then taking the trace of $\hat{\rho}(t)$ with respect to all subsystem matrix elements within the system but those of subsystem ν. For our oscillator system under study, the state evolution is given by

$$|\psi(t)\rangle \;=\; \sum_{n_1,n_2=0}^{\infty} c_{n_1,n_2}\;\, |n_1, n_2\rangle_H\;\, e^{-\frac{i}{\hbar}E_{n_1,n_2}\,t}\,, \tag{13}$$

where $c_{n_1,n_2} \;=\; \langle\psi(0)|\; n_1, n_2\rangle_H^*$. Therefore, in constructing $\hat{\rho} \;=\; |\psi\rangle\,\langle\psi|$ from (13), and taking the trace of $\hat{\rho}$ with respect to oscillator 2, we formally acquire the reduced density matrix for oscillator 1 as

$$\hat{\rho}_{kl}^{(1)}(t) \;=\; \mathrm{Tr}_2\,\hat{\rho}(t) \;=\; \sum_{s=0}^{\infty}\, \{\hat{\rho}(t)\}_{ks;ls}\;, \tag{14}$$

where $\{\hat{\rho}(t)\}_{ks;ls} \equiv \langle s_2|\,\langle k_1|\,\hat{\rho}(t)\,|l_1\rangle\,|s_2\rangle$. Using $|\psi(t)\rangle$ in eq. (13), we explicitly obtain

$$\hat{\rho}_{kl}^{(1)}(t) \;=\; \sum_{n_1,n_2}\sum_{n_1',n_2'} c_{n_1,n_2}\cdot c_{n_1',n_2'}^*\cdot e^{-\frac{i}{\hbar}(E_{n_1,n_2}-E_{n_1',n_2'})\,t}\,\cdot\,\sum_{s}\,\langle s_2|\,\langle k_1|\; e^{-i\gamma\hat{\mathcal{J}}_y/\hbar}\;|n_1\rangle\,|n_2\rangle\cdot\langle n_2'|\,\langle n_1'|\; e^{i\gamma\hat{\mathcal{J}}_y/\hbar}\;|l_1\rangle\,|s_2\rangle\,, \tag{15}$$

where $\hat{\mathcal{J}}_y = \hat{\mathcal{J}}_y(0)$, and $|n_1, n_2\rangle_H$ from eq. (11) has been used in eq. (13) to obtain the explicit result.

Now, in transforming eq. (15) to the coupled-boson representation discussed in eq. (2), we relabel $|n_1\rangle|n_2\rangle$ and $|n_1'\rangle|n_2'\rangle$ in terms of the eigenstates of $\hat{\mathcal{J}}_z$, namely, $|j;m\rangle$ and $|j';m'\rangle$, respectively, where $j = \frac{1}{2}(n_1+n_2)$, $m = \frac{1}{2}(n_1-n_2)$ and $j' = \frac{1}{2}(n_1'+n_2')$, $m' = \frac{1}{2}(n_1'-n_2')$. Since the rotation with respect to $\hat{\mathcal{J}}_y$ conserves the quantum number j as noted in eq. (4), we also find that

$$k + s = 2j\,;\; l + s = 2j'\,. \tag{16}$$

Taking into account that $m = -j, -j+1, \cdots, j$ and $m' = -j', -j'+1, \cdots, j'$ for a given j, j', respectively, it follows that eq. (15) can be reexpressed as

$$\hat{\rho}_{kl}^{(1)}(t) = \sum_{j,j'=0}^{\infty} \sum_{m=-j}^{j} \sum_{m'=-j'}^{j'} c_{j,m} \cdot c_{j',m'}^* \cdot e^{-\frac{i}{\hbar}(E_{j+m,j-m} - E_{j'+m',j'-m'})\,t} \cdot \langle j; k-j|\, e^{-i\gamma\hat{\mathcal{J}}_y/\hbar}\,|j;m\rangle\, \langle j';m'|\, e^{i\gamma\hat{\mathcal{J}}_y/\hbar}\,|j';l-j'\rangle \tag{17}$$

where $k = 0, 1, \cdots, 2j$ and $l = 0, 1, \cdots, 2j'$, respectively. Then, in defining $\mathbf{k} \equiv k - j$ and $\mathbf{l} \equiv l - j$ we relabel $\hat{\rho}_{kl}^{(1)}$ in terms of $\hat{\rho}_{\mathbf{kl}}^{(1)}$, where $\mathbf{k} = -j, -j+1, \cdots, j$ and $\mathbf{l} = -j', -j'+1, \cdots, j'$, respectively. This clearly shows that two coupled harmonic oscillators can naturally be treated as an angular momentum oscillator. We now make use of the coupled-boson algebra to evaluate the reduced density matrix $\hat{\rho}_{\mathbf{kl}}^{(1)}(t)$ below for two physically interesting, transparent, and mathematically manageable initial states, $|\psi(0)\rangle$, to illustrate the entanglement measure $\mathcal{M}_{|\psi(t)\rangle}$, namely, the case of a simple product state, $|N_1\rangle\,|N_2\rangle$ (case I), and the case of an eigenstate of the exact Hamiltonian in eq. (3), $|N_1, N_2\rangle_H$ of eq. (11) (case II). In terms of physical interest, the simple product state in case I is an initial condition which inherently assumes no entanglement at time zero, whereas case II, using the exact eigenstate of the Hamiltonian, implicitly includes entanglement.

Let us begin with the case I: $|\psi(0)\rangle = |N_1\rangle\,|N_2\rangle$. In the coupled-boson representation, this state becomes $|J;M\rangle$ where $J = \frac{1}{2}(N_1+N_2)$ and $M = \frac{1}{2}(N_1-N_2)$. We now have $c_{j,m} = \langle J;M|\, j;m\rangle_H^* \cdot \delta_{jJ}$ and $c_{j',m'}^* = \langle J;M|\, j';m'\rangle_H \cdot \delta_{j'J}$ in eq. (17) which then reduces to the diagonal form,

$$\hat{\rho}_{\mathbf{kl}}^{(1)}(t) = \delta_{\mathbf{kl}} \sum_{m,m'=-J}^{J} \langle J;M|\, J;m\rangle_H^*\, \langle J;M|\, J;m'\rangle_H \times \langle J;\mathbf{k}|\, J;m\rangle_H\, \langle J;\mathbf{l}|\, J;m'\rangle_H^*\, e^{i(m-m')\bar{\omega}t}\,, \tag{18}$$

where $\delta_{\mathbf{kl}}$ results from eq. (16) with $j = j' = J$. In eq. (18), all the matrix elements can be written in terms of $\langle J;\mu_2|\, J;\mu_1\rangle_H \equiv U_{\mu_1\mu_2}^{(J)}(\gamma)$ with $(\mu_1 = m, m')$ and $(\mu_2 = M, \mathbf{k}, \mathbf{l})$, where $U_{\mu_1\mu_2}^{(J)}(\gamma)$ is formally explicitly expressed as[4]

$$U_{\mu_1\mu_2}^{(J)}(\gamma) = \sqrt{\tfrac{(J+\mu_2)!}{(J-\mu_2)!}}\, \frac{(\sin\frac{\gamma}{2})^{\mu_1-\mu_2}\,(\cos\frac{\gamma}{2})^{-\mu_1-\mu_2}}{(2^{J+\mu_2})\,\sqrt{(J+\mu_1)!\,(J-\mu_1)!}} \times \left(\tfrac{d}{d\cos\gamma}\right)^{J-\mu_2} \left\{(\cos\gamma+1)^{J+\mu_1}\,(\cos\gamma-1)^{J-\mu_1}\right\}; \tag{19}$$

here we see that $U_{\mu_2\mu_1}^{(J)}(\gamma) = U_{\mu_1\mu_2}^{(J)}(-\gamma)$, and $\{U_{\mu_1\mu_2}^{(J)}(\gamma)\}^* = U_{\mu_1\mu_2}^{(J)}(\gamma)$. Particularly for the case that $\mu_2 = J$, this reduces further to

$$U_{\mu J}^{(J)}(\gamma) = \sqrt{\tfrac{(2J)!}{(J+\mu)!\,(J-\mu)!}}\, \left(-\sin\tfrac{\gamma}{2}\right)^{J-\mu} \left(\cos\tfrac{\gamma}{2}\right)^{J+\mu}\,, \tag{20}$$

which will be used later. Equivalently, $U_{\mu_1\mu_2}^{(J)}(\gamma)$ can be expressed in terms of the tabulated Jacobi polynomial[11] $\mathcal{P}_n^{(\alpha,\beta)}(x)$ in the form

$$U_{\mu_1\mu_2}^{(J)}(\gamma) = (-1)^{\mu_1+\mu_2} \sqrt{\tfrac{(J+\mu_2)!\,(J-\mu_2)!}{(J+\mu_1)!\,(J-\mu_1)!}}\, \left(\sin\tfrac{\gamma}{2}\right)^{\mu_2-\mu_1} \times \left(\cos\tfrac{\gamma}{2}\right)^{\mu_1+\mu_2} \cdot \left\{\mathcal{P}_{J-\mu_2}^{(\mu_2-\mu_1,\mu_1+\mu_2)}(\cos\gamma)\right\}. \tag{21}$$

Reexpressing $\hat{\rho}_{\mathbf{kl}}^{(1)}$ in eq. (18) in terms of $U_{\mu_1\mu_2}^{(J)}(\gamma)$, we obtain

$$\hat{\rho}_{\mathbf{kl}}^{(1)}(t) = \delta_{\mathbf{kl}} \sum_{m,m'=-J}^{J} \{U_{mM}^{(J)}(\gamma)\}^* \cdot U_{m'M}^{(J)}(\gamma) \cdot U_{m\mathbf{k}}^{(J)}(\gamma) \cdot \{U_{m'\mathbf{l}}^{(J)}(\gamma)\}^* \cdot e^{i(m-m')\bar{\omega}t} = \delta_{\mathbf{kl}} \cdot f_{\mathbf{l}}^{(1)}(t) \cdot \{f_{\mathbf{k}}^{(1)}(t)\}^* \equiv \delta_{\mathbf{kl}} \cdot |f_{\mathbf{k}}^{(1)}(t)|^2\,, \tag{22}$$

where

$$f_{\mathbf{k}}^{(1)}(t) \equiv \sum_{m=-J}^{J} U_{mM}^{(J)}(\gamma) \cdot U_{m\mathbf{k}}^{(J)}(\gamma) \cdot e^{-im\bar{\omega}t} \tag{23}$$

with the property that $f^{(1)}_{\mathbf{k}}(0) = \delta_{M\mathbf{k}}$ [here, we used the relation that $\{U^{(J)}_{mM}(\gamma)\}^* = U^{(J)}_{mM}(\gamma)$]; $f^{(1)}_{\mathbf{k}}(t)$ can be explicitly evaluated for any set of the harmonic oscillator initial condition numbers, $\{N_1,\, N_2\}$. We refer the reader to the Appendix A for the explicit expression of eq. (23) obtained from the substitution of eq. (21), which will be useful below. From (22) we obtain the measure of entanglement given by

$$\mathcal{M}_{|\psi(t)\rangle} \;=\; 1 - \sum_{\mathbf{k}=-J}^{J} |f^{(1)}_{\mathbf{k}}(t)|^4 \;\geq\; 0\,. \tag{24}$$

We note generally that $f^{(1)}_{\mathbf{k}}$ in eq. (23) is a time-dependent oscillatory function with a modal distribution coefficient $U^{(J)}_{mM}(\gamma)\cdot U^{(J)}_{m\mathbf{k}}(\gamma)$ governing the oscillatory strength. This clearly influences the measure of entanglement as noted in eq. (24) and as seen in Fig. 1 for specific values of J.

Let us now apply eqs. (22) and (24) for specific values of J. For $J = \frac{1}{2}$, the total occupation number $N_1 + N_2 = 1$, there are two possible simple product states $\left|\frac{1}{2};\frac{1}{2}\right\rangle, \left|\frac{1}{2};-\frac{1}{2}\right\rangle$ describing this initial condition. We can easily evaluate $\hat{\rho}^{(1)}(t)$: for $|\psi_1(0)\rangle = \left|\frac{1}{2};\frac{1}{2}\right\rangle$, from eqs. (20) and (23) we obtain $f^{(1)}_{-\frac{1}{2}}(t) = \cos\frac{\bar{\omega}t}{2} + i\,(\cos\gamma)\,\sin\frac{\bar{\omega}t}{2}$, and then the 2×2 diagonal matrix from eq. (22),

$$\hat{\rho}^{(1)}_{-\frac{1}{2},-\frac{1}{2}} \;=\; \tfrac{1}{2}\,(\sin\gamma)^2\,\left\{1-\cos\left(t\bar{\omega}\right)\right\}\;;\;\hat{\rho}^{(1)}_{\frac{1}{2},\frac{1}{2}} \;=\; 1 \,-\, \hat{\rho}^{(1)}_{-\frac{1}{2},-\frac{1}{2}}\,. \tag{25}$$

Similarly, for $|\psi_2(0)\rangle = \left|\frac{1}{2};-\frac{1}{2}\right\rangle$, we acquire the diagonal matrix $\hat{\rho}^{(1)}_2(t)$ whose elements are exchanged from eq. (25). Due to the fact that $\mathcal{P}[\hat{\rho}^{(1)}_1(t)] = \mathcal{P}[\hat{\rho}^{(1)}_2(t)] \leq 1$, the reduced density matrices, $\hat{\rho}^{(1)}_1(t)$ and $\hat{\rho}^{(1)}_2(t)$, represent mixed states, respectively, with both directly reflecting the appearance of entanglement between the two coupled oscillators; this is noted in Fig. 1 with the properties of $\mathcal{M}_{|\psi_1(t)\rangle}$ and $\mathcal{M}_{|\psi_2(t)\rangle}$ as given in eq. (24). This appearance of entanglement in the time evolution from the disentangled initial product state for each case is clearly attributed to the interaction Hamiltonian in (1), i.e., $\hbar\,\kappa\,(\hat{a}^\dagger_1\,\hat{a}_2 + \hat{a}_1\,\hat{a}^\dagger_2)$. Likewise, for $J = 1, \frac{3}{2}, 5$ (or $N_1 + N_2 = 2, 3, 10$), we also obtain the entanglement in the time evolution resulting from the interaction Hamiltonian. Due to their complex forms of eqs. (22) and (24) [see also (A1) and (A2)], respectively, we simply plot the exact numerical results of $\mathcal{M}_{|\psi(t)\rangle}$ for $N = N_1 + N_2$ with $|\psi(0)\rangle = |N\rangle|0\rangle = |J;J\rangle$ (see Fig. 1). We clearly see that $\mathcal{M}_{|\psi(t)\rangle}$ increases with N (or J).

Two points deserve comment here. First, we see that for a given total occupation number $N = 2J$ [constant of motion; cf. (4)], $\mathcal{M}_{|\psi(t)\rangle}$ is periodic (see Fig. 1), as is the reduced density matrix $\hat{\rho}^{(1)}(t)$ due to the clear oscillator dependence on time as seen in $f^{(1)}_{\mathbf{k}}$ in (23); on the other hand, this is not the case for a single oscillator coupled with a thermal reservoir modeled by a sea of infinite oscillators[12], for then, dissipative decay of its occupation number into the the reservoir is effective. Second, in calculating $\hat{\rho}^{(2)}(t) = \mathrm{Tr}_1\,\hat{\rho}(t)$ and comparing results to the corresponding expressions in (14) - (18), one easily arrives at the fact that $\hat{\rho}^{(2)}_{\mathbf{kl}} = \hat{\rho}^{(1)}_{-\mathbf{k},-\mathbf{l}}$ for any J, which confirms the relationship valid for any system state $|\psi(t)\rangle$ that $\mathcal{P}[\hat{\rho}^{(2)}] = \mathcal{P}[\hat{\rho}^{(1)}]$, thus indicating the equality of mixture.

Next, we consider case II, in which the initial state is given by an eigenstate of the exact Hamiltonian in eq. (3), $|N_1,N_2\rangle_H = |J;M\rangle_H$ with $J = \frac{1}{2}(N_1+N_2)$ and $M = \frac{1}{2}(N_1-N_2)$. From eq. (17), we easily obtain the reduced density matrix and the measure of entanglement,

$$\hat{\rho}^{(1)}_{\mathbf{kl}} \;=\; \delta_{\mathbf{kl}}\cdot\left|\left\langle J;M\left|e^{i\gamma\hat{J}_y/\hbar}\right|J;\mathbf{k}\right\rangle\right|^2 \;=\; \delta_{\mathbf{kl}}\cdot|f^{(2)}_{\mathbf{k}}|^2\,, \tag{26}$$

where

$$f^{(2)}_{\mathbf{k}} \;\equiv\; \hat{U}^{(J)}_{M\mathbf{k}}(\gamma)\,. \tag{27}$$

Here, $f^{(2)}_{\mathbf{k}}$ is time-independent, whereas $f^{(1)}_{\mathbf{k}}(t)$ from eq. (23) is not as obtained in the previous case. Also, we can easily show that $\hat{\rho}^{(2)}_{\mathbf{kl}} = \hat{\rho}^{(1)}_{-\mathbf{k},-\mathbf{l}}$. Then, as in eq. (24), we have

$$\mathcal{M}_{|\psi(t)\rangle} \;=\; 1 \,-\, \sum_{\mathbf{k}=-J}^{J} |f^{(2)}_{\mathbf{k}}|^4\,. \tag{28}$$

Particularly for $M = J$, namely $N_2 = 0$, from eq. (20) with $U^{(J)}_{J\mathbf{k}}(\gamma) = U^{(J)}_{\mathbf{k}J}(-\gamma)$, we easily find that

$$f^{(2)}_{\mathbf{k}} \;=\; U^{(J)}_{J\mathbf{k}}(\gamma) \;=\; \sqrt{\tfrac{(2J)!}{k!\,(2J-k)!}}\;\left(\cos\tfrac{\gamma}{2}\right)^k\;\left(\sin\tfrac{\gamma}{2}\right)^{2J-k}\,, \tag{29}$$

where $k = J + \mathbf{k}$. Therefore, from eq. (26) we arrive at the fact that the occupation number k in oscillator 1 is described by the binomial distribution $\mathcal{B}(k;\, 2J, p_1)$ with a trial probability $p_1 = (\cos\frac{\gamma}{2})^2$, which remains unchanged in the time evolution. Along the same line, for oscillator 2, we also have

$$f_{\mathbf{k}}^{(2)} \;=\; U_{J\mathbf{k}}^{(J)}(\gamma) \;=\; \sqrt{\tfrac{(2J)!}{k!\,(2J-k)!}}\, \left(\sin\tfrac{\gamma}{2}\right)^{k}\, \left(\cos\tfrac{\gamma}{2}\right)^{2J-k} \tag{30}$$

and, from eq. (26), the binomial distribution $\mathcal{B}(k;\, 2J, p_2)$ with $p_2 = (\sin\frac{\gamma}{2})^2$ for the occupation number k in oscillator 2. These reduced density matrices, $\hat{\rho}^{(1)}$ and $\hat{\rho}^{(2)}$, represent mixed states for $\gamma \neq 0$, respectively, due to the fact that $\mathcal{P}[\hat{\rho}_1^{(1)}] = \mathcal{P}[\hat{\rho}_2^{(1)}] \leq 1$, thus indicating the entanglement between the two linear oscillators, $0 \leq \mathcal{M}_{|\psi(t)\rangle}$ obtained from eq. (28). Furthermore, as in the previous case, $\mathcal{M}_{|\psi(t)\rangle}$ increases with J (see Fig. 2); it is noted in the figure that $\mathcal{M}_{|\psi(t)\rangle} \to 1$ as $J \to \infty$.

3. INTERACTION BETWEEN A LINEAR AND A NON-LINEAR OSCILLATOR

3.1. Quantum-classical behaviors in the Heisenberg picture

The second coupled system under investigation consists of a linear oscillator and a (non-linear) angular momentum oscillator. This system approximates the interaction between a field mode and an atomic N-level system under idealized conditions resulting from neglecting dissipation and the coupling between other surrounding atomic systems. Following the seminal work of Ref. 3, which considered the analysis quantum-mechanically and also classically to compare one case with the other systematically, it is convenient to employ, for the linear oscillator, dimensionless coordinate $\mathtt{x}$ and momentum $\mathtt{p}$, where $\mathtt{x} \equiv \sqrt{\frac{m\omega}{\hbar}}\, x$ and $\mathtt{p} \equiv \frac{1}{\sqrt{m\hbar\omega}}\, p$, satisfying $[\mathtt{x}, \mathtt{p}]_{\mathtt{p}} = i$, and, for the angular momentum oscillator, dimensionless angular momentum variables $\mathtt{j}_x$, $\mathtt{j}_y$, $\mathtt{j}_z$ with $[\mathtt{j}_r, \mathtt{j}_s]_{\mathtt{p}} = i\,\epsilon_{rst}\,\mathtt{j}_t$, where the brackets $[\,,\,]_{\mathtt{p}}$ represent the commutators for the quantum-mechanical description, and i times the Poisson brackets for the classical description, respectively.

The Hamiltonians of the two individual oscillators are then expressed as

$$H_1 \;=\; \frac{1}{2}\,\hbar\,\omega_1\left(\mathtt{x}_1^2 \,+\, \mathtt{p}_1^2\right)\;,\; H_2 \;=\; \hbar\,\omega_2\,(\mathtt{j}_z)_2\;, \tag{31}$$

respectively, and the interaction Hamiltonian is given in the *rotating wave approximation* by

$$H_{12} \;=\; \hbar\,\kappa\,\Big(a_1\,(\mathtt{j}_+)_2 \,+\, a_1^\dagger\,(\mathtt{j}_-)_2\Big)\;, \tag{32}$$

leading to the total Hamiltonian

$$H \;=\; H_0 \,+\, H_{12}\,;\; H_0 \;\equiv\; H_1 \,+\, H_2\,. \tag{33}$$

Here, we have

$$a_1 \equiv \tfrac{1}{\sqrt{2}}\,(\mathtt{x}_1 + i\,\mathtt{p}_1)\;,\; a_1^\dagger \equiv \tfrac{1}{\sqrt{2}}\,(\mathtt{x}_1 - i\,\mathtt{p}_1)\;,\; (\mathtt{j}_+)_2 \equiv \tfrac{1}{\sqrt{2}}\left\{(\mathtt{j}_x)_2 + i\,(\mathtt{j}_y)_2\right\}\;,\; (\mathtt{j}_-)_2 \equiv \tfrac{1}{\sqrt{2}}\left\{(\mathtt{j}_x)_2 - i\,(\mathtt{j}_y)_2\right\} \tag{34}$$

(classically, Hermitian conjugation corresponds to complex conjugation). These non-Hermitian variables and $(\mathtt{j}_z)_2$ clearly obey

$$\left[a_1,\, a_1^\dagger\right]_{\mathtt{p}} \;=\; 1\;,\; [(\mathtt{j}_+)_2\,,\,(\mathtt{j}_-)_2]_{\mathtt{p}} \;=\; (\mathtt{j}_z)_2\;,\; [(\mathtt{j}_+)_2\,,\,(\mathtt{j}_z)_2]_{\mathtt{p}} \;=\; -\,(\mathtt{j}_+)_2\;,\; [(\mathtt{j}_-)_2\,,\,(\mathtt{j}_z)_2]_{\mathtt{p}} \;=\; (\mathtt{j}_-)_2 \tag{35}$$

with all other brackets vanishing. From now on, let us restrict ourselves to the resonant case, $\omega_1 = \omega_2 \equiv \omega$. In order to make later calculations simpler, we then introduce the reduced variables[13] specified, with the aid of the identity (5) [for the classical description, the commutators therein have to be obviously replaced by i times Poisson brackets] and the rules in (35), by

$$A_1 \;\equiv\; e^{-iH_0\,t/\hbar}\,a_1\,e^{iH_0\,t/\hbar} \;=\; a_1\,e^{i\omega t}\;,\; A_1^\dagger \;\equiv\; e^{-iH_0\,t/\hbar}\,a_1^\dagger\,e^{iH_0\,t/\hbar} \;=\; a_1^\dagger\,e^{-i\omega t} \tag{36}$$

and, similarly,

$$(J_+)_2 \;=\; (\mathtt{j}_+)_2\,e^{-i\omega t}\;,\; (J_-)_2 \;=\; (\mathtt{j}_-)_2\,e^{i\omega t}\;,\; (J_z)_2 \;=\; (\mathtt{j}_z)_2\;, \tag{37}$$

respectively. These reduced variables are easily seen to satisfy the same bracket relations as the corresponding unreduced variables, namely,

$$\left[A_1,\, A_1^\dagger\right]_{\mathsf{p}} \;=\; 1\;,\; \left[(J_+)_2\,,\,(J_-)_2\right]_{\mathsf{p}} \;=\; (J_z)_2\;,\; \left[(J_+)_2\,,\,(J_z)_2\right]_{\mathsf{p}} \;=\; -\,(J_+)_2\;,\; \left[(J_-)_2\,,\,(J_z)_2\right]_{\mathsf{p}} \;=\; (J_-)_2\,. \tag{38}$$

The interaction Hamiltonian in (32) is now rewritten as

$$H_{12} \;=\; \hbar\,\kappa\,\left\{A_1\,(J_+)_2 \;+\; A_1^\dagger\,(J_-)_2\right\}\,, \tag{39}$$

and the equations of motion in terms of the reduced variables will be given by

$$i\,\hbar\,\dot{O}(t) \;=\; [O(t),\,H_{12}]_{\mathsf{p}}\,, \tag{40}$$

where O stands for any of the above reduced variables.

From eqs. (38), (40) it follows that

$$\begin{gathered} \dot{A}_1 \;=\; -i\,\kappa\,(J_-)_2\;,\; \dot{A}_1^\dagger \;=\; i\,\kappa\,(J_+)_2\,, \\ (\dot{J}_+)_2 \;=\; -i\,\kappa\,A_1^\dagger\,(J_z)_2\;,\; (\dot{J}_-)_2 \;=\; i\,\kappa\,A_1\,(J_z)_2\;,\; (\dot{J}_z)_2 \;=\; -i\,\kappa\,\left\{A_1\,(J_+)_2 \;-\; A_1^\dagger\,(J_-)_2\right\} \end{gathered} \tag{41}$$

and then

$$(\ddot{J}_z)_2 \;=\; -\kappa^2\,\left\{(J_-)_2\,(J_+)_2 \;+\; (J_+)_2\,(J_-)_2 \;+\; \left(A_1\,A_1^\dagger \;+\; A_1^\dagger\,A_1\right)\,(J_z)_2\right\} \;=\; -2\,\kappa^2\,\left\{(J_-)_2\,(J_+)_2 \;+\; A_1\,A_1^\dagger\,(J_z)_2\right\}, \tag{42}$$

which, as shown previously,[3] hold both quantum-mechanically in the Heisenberg picture and classically on the basis of the Poisson bracket in phase space.

From the equation of motion in eq. (40) it turns out that the sum of the (dimensionless) energy of both oscillators, $\mathtt{E} = n_1 + (J_z)_2$ with $n_1 \equiv A_1^\dagger\,A_1$ (without the zero-point energy of the linear oscillator in the quantum-mechanical description), and the interaction Hamiltonian H_{12} are constants of motion, respectively, which are determined by a given initial state. As well, the square of a given total angular momentum of the angular momentum oscillator, $\mathbf{J}^2 = (J_x)_2^2 + (J_y)_2^2 + (J_z)_2^2 = (J_+)_2\,(J_-)_2 + (J_-)_2\,(J_+)_2 + (J_z)_2^2$, is also a constant of motion. Then, substituting the expressions

$$A_1\,A_1^\dagger \;=\; \mathtt{E} \,-\, (J_z)_2 \,+\, \left[A_1,\,A_1^\dagger\right]\;,\;\; 2\,(J_-)_2\,(J_+)_2 \;=\; \mathbf{J}^2 \,-\, (J_z)_2^2 \,-\, [(J_+)_2\,,\,(J_-)_2] \tag{43}$$

(note that there are no subscripts p of the brackets) into eq. (42), we obtain

$$(\ddot{J}_z)_2 \;=\; \kappa^2\left\{3\,(J_z)_2^2 \,-\, 2\left(\mathtt{E} \,+\, \left[A_1,\,A_1^\dagger\right]\right)\,(J_z)_2 \,+\, [(J_+)_2\,,\,(J_-)_2] \,-\, \mathbf{J}^2\right\}\,. \tag{44}$$

Let us introduce the notation

$$\left[A_1,\,A_1^\dagger\right] \;=\; \lambda_1\;,\;\; [(J_+)_2\,,\,(J_-)_2] \;=\; \lambda_2\,(J_z)_2\,, \tag{45}$$

where $\lambda_k = 1, 0$ with $k = 1, 2$ correspond to the quantum-mechanical and the classical description of each oscillator, respectively. Eq. (44) is now reduced to

$$(\ddot{J}_z)_2 \;=\; \kappa^2\left\{3\,(J_z)_2^2 \,-\, (2\,\mathtt{E} \,+\, 2\,\lambda_1 \,-\, \lambda_2)\,(J_z)_2 \,-\, j\,(j \,+\, \lambda_2)\right\}\,, \tag{46}$$

where the constant of motion $\mathbf{J}^2 \doteq j\,(j + \lambda_2)$ with j being classically ($\lambda_2 = 0$) the total (continuous) angular momentum and quantum-mechanically ($\lambda_2 = 1$) the corresponding quantum number. Here, $(J_z)_2$ is, clearly, an operator for $\lambda_2 = 1$, with the property $\langle (J_z)_2^2\rangle \neq \langle (J_z)_2\rangle^2$ in general. Eq. (46) is a non-linear differential equation for $(J_z)_2$ in both the classical and quantum-mechanical description. Since $[(J_z)_2\,,\mathtt{E}]_{\mathsf{p}} = [(\dot{J}_z)_2,\mathtt{E}]_{\mathsf{p}} = 0$ (note that $\mathtt{E} = n_1 + (J_z)_2$), it follows that in the quantum-mechanical consideration of eq. (46) the constant of motion $\mathtt{E}$ can be treated as a c-number.

We now consider eq. (46) with two interesting initial states. First, for the ground state with $n_1 = 0$ and $(J_z)_2 = -j$, then $\mathtt{E} \equiv \mathtt{E}_g = -j$; it immediately follows that eq. (46), at $t = 0$, becomes

$$(\ddot{J}_z)_2\Big|_{t=0} \;=\; 2\,\kappa^2\,(\lambda_1 \,-\, \lambda_2)\,j\,. \tag{47}$$

Quantum-mechanically, this state corresponds obviously to $|\psi_g\rangle \equiv |0\rangle_1\,|j,-j\rangle_2$. From the fact that each oscillator with the minimum energy cannot give up further energy, we necessarily have $(\ddot{J}_z)_2 = 0$ here, which indicates that only $\lambda_1 \equiv \lambda_2 = 0, 1$ simultaneously are physically allowed. For the quantum description, the zero-point fluctuation $\lambda_1 = 1$ would yield a positive force on the angular momentum oscillator, and the fluctuation $\lambda_2 = 1$ of the angular momentum oscillator would lead to energy transfer into the linear oscillator; however, the two formal processes are always cancelled such that they are not part of the real (or measurable) physical processes of energy transfer.[14] Therefore, the "semiclassical" option with $\lambda_1 = 0$ and $\lambda_2 = 1$, and vice versa, is not physically admissible for the ground state $|\psi_g\rangle$, and is thus inappropriate for the system analysis.

Another initial state under investigation is the state $\mathtt{E} = j$ with $n_1 = 0$ and $(J_z)_2 = j$, quantum-mechanically $|\psi_{\mathbf{J}}\rangle \equiv |0\rangle_1\,|j,j\rangle_2$. Then, eq. (46), at $t = 0$, reduces to

$$(\ddot{J}_z)_2\Big|_{t=0} \;=\; -2\,\kappa^2\,\lambda_1\,j\,, \tag{48}$$

independent of λ_2! This demonstrates that only for the quantum description of the linear oscillator ($\lambda_1 = 1$) is spontaneous emission available, which arises from the zero-point fluctuation. Interestingly, it points out that for $\lambda_1 = 0$, there exists an unstable equilibrium leading to the absence of spontaneous emission; the unstable equilibrium in the classical description ($\lambda_1 = \lambda_2 = 0$) and the presence of spontaneous emission for $\lambda_1 = 1$ were thoroughly discussed in Ref. 3. Here, we connect the spontaneous emission process with the notion of entanglement. In considering the time evolution of the wave-function for the spontaneous emission in the Schrödinger picture, $|\psi(t)\rangle = e^{-iHt/\hbar}\,|0\rangle_1\,|j,j\rangle_2$ with the Hamiltonian H in eq. (33), we now necessarily have the entangled state for $t \to 0^+$, i.e., early time beyond the initial time, which is given by

$$|\psi(t)\rangle \;=\; |0\rangle_1\,|j,j\rangle_2 \,-\, i\,t\,\left\{\omega\left(j + \tfrac{1}{2}\right)|0\rangle_1\,|j,j\rangle_2 \,-\, \kappa\sqrt{j}\,|1\rangle_1\,|j,j-1\rangle_2\right\} \,+\, O(t^2)\,. \tag{49}$$

This entanglement is explicitly shown in a compact form for the two-level case ($j = \frac{1}{2}$) later in eq. (58) and in the more complex form for the three-level case ($j = 1$) in (71). Therefore, for the quantum description $\lambda_1 = \lambda_2 = 1$, the entanglement is always temporally present in the spontaneous emission process. Here, interestingly enough, we still have the spontaneous emission for large enough j, which immediately leads to the fact that the entanglement persists even as the *coupled* angular momentum oscillator is taken to the limit of a large number of levels, a limit which would go over to the classical limit for an *uncoupled* angular momentum oscillator. In essence, the state $|\psi_{\mathbf{J}}\rangle$ of the uppermost excited state of a j-level angular momentum oscillator is always coupled to the vacuum state of the linear oscillator.

From eqs. (39), (41) we also obtain

$$(\dot{J}_z)_2^2 \,+\, K^2 \;=\; 2\,\kappa^2\,\left\{A_1\,A_1^\dagger\,(J_+)_2\,(J_-)_2 \,+\, A_1^\dagger\,A_1\,(J_-)_2\,(J_+)_2\right\}\,, \tag{50}$$

where $K \equiv H_{12}/\hbar$, and subsequently, with the aid of eqs. (43), (45), the expression

$$(\dot{J}_z)_2^2 \;=\; 2\,\kappa^2\left[\left\{(J_z)_2 \,-\, \left(\mathtt{E} \,+\, \tfrac{\lambda_1}{2}\right)\right\}\left\{(J_z)_2^2 \,-\, \left(j\,(j\,+\,\lambda_2) \,-\, \tfrac{1}{2}\lambda_1\,\lambda_2\right)\right\}\right] \,+\, \left(\mathtt{E} \,+\, \tfrac{\lambda_1}{2}\right)\lambda_1\,\lambda_2\,\kappa^2 \,-\, K^2\,. \tag{51}$$

In Ref. 3, it was indicated that in the classical description ($\lambda_1 = \lambda_2 = 0$) eq. (51) has the form of an equation for the vertical position J_z of a classical spherical pendulum, where the constant of motion K represents an angular momentum about the vertical axis through the center. It turns out that for $n_1 = 0$ and $(J_z)_2 = j$ we get $K = 0$ from (39) with $A_1 = \sqrt{n_1}\,e^{i\,\theta_1}$, $A_1^* = \sqrt{n_1}\,e^{-i\,\theta_1}$ and $J_\pm = \sqrt{(\mathbf{J}^2 - (J_z)^2)/2}\,e^{\pm i\,\theta_2}$ (note that in the quantum description the constant of motion $\langle\psi_{\mathbf{J}}|\,K^2\,|\psi_{\mathbf{J}}\rangle \neq 0$) and then $(\dot{J}_z)_2^2 = 0$ from (51). With $(\ddot{J}_z)_2 = 0$ in eq. (48), this reveals that the initial state ($n_1 = 0$, $(J_z)_2 = j$) corresponds to an unstable equilibrium of the pendulum.

Here, it is also worthwhile pointing out here that the semiclassical treatment of Jaynes-Cummings[15] reproduces this spontaneous emission for $j = \frac{1}{2}$ only, with the same decay rate as that given by the quantum-mechanical description; this is accomplished by adding a phenomenological damping term to the classical equation of motion of the linear oscillator which would result from the interaction with the instantaneous expectation value of the dipole moment of the angular-momentum oscillator. Therefore, the Jaynes-Cummings model just provides an "artificial" picture of the spontaneous emission without describing the actual physical processes involved therein (e.g., entanglement), and also without offering a direct way for its extension to more than the two-level case ($j \geq 1$).

3.2. Appearance of the entanglement in the Schrödinger picture

In order to study the appearance of the entanglement between the linear and the angular momentum oscillators in the quantum description, we discuss the state evolution of the total system in the Schrödinger picture given by

$$\hat{H} \;=\; \hbar\,\omega\,(\hat{n}_1 + \tfrac{1}{2}) \,+\, \hbar\,\omega\,\hat{J}_z \,+\, \hbar\,\kappa\,(\hat{A}_1\,\hat{J}_+ \,+\, \hat{A}_1^\dagger\,\hat{J}_-) \tag{52}$$

from eqs. (31) - (33) with the reduced variables in eqs. (36) - (37) for the resonant case, $\omega_1 = \omega_2 \equiv \omega$. We will explicitly consider below the measure of entanglement for the cases of $j = \frac{1}{2}, 1, \frac{3}{2}$ of the angular momentum oscillator. To this end, as will be seen, the $(2j+1) \times (2j+1)$ irreducible matrix representation for the total system is employed, which is also systematically applicable for $j \geq 2$. Here, we assume that the given total energy (a constant of motion) is given by $\mathtt{E} \equiv n_1 + J_z = n_1 + j \geq j$.

3.2.1. *Case* $j = \frac{1}{2}$

We first consider the case of $j = \frac{1}{2}$, namely, a two-level angular momentum oscillator interacting with a linear oscillator. For the total energy $\mathtt{E} = n_1 + j = n_1 + \frac{1}{2}$, the basis $\{|n_1+1, -\frac{1}{2}\rangle, |n_1, \frac{1}{2}\rangle\}$ allows us to rewrite eq. (52) in explicit operator form[16] as

$$\begin{aligned}\hat{H} \;=\; & (n_1+1)\,\hbar\,\omega\,\left\{\,\left|n_1+1,-\tfrac{1}{2}\right\rangle\left\langle n_1+1,-\tfrac{1}{2}\right| \,+\, \left|n_1,\tfrac{1}{2}\right\rangle\left\langle n_1,\tfrac{1}{2}\right|\,\right\} \,+ \\ & \tfrac{1}{\sqrt{2}}\,\hbar\,\kappa\,\sqrt{n_1+1}\,\left\{\,\left|n_1+1,-\tfrac{1}{2}\right\rangle\left\langle n_1,\tfrac{1}{2}\right| \,+\, \left|n_1,\tfrac{1}{2}\right\rangle\left\langle n_1+1,-\tfrac{1}{2}\right|\,\right\}\,.\end{aligned} \tag{53}$$

This can be written in a 2×2 matrix form as

$$\hat{H} \;=\; (n_1+1)\,\hbar\,\omega\,\mathbb{1}_2 \,+\, \tfrac{1}{\sqrt{2}}\,\hbar\,\kappa\,\sqrt{n_1+1}\,\hat{\sigma}_x\,, \tag{54}$$

where $\hat{\sigma}_x = 2\hat{J}_x$ denotes the Pauli matrix. Using the identity,

$$e^{i\alpha\hat{\sigma}_y}\,\hat{\sigma}_x\,e^{-i\alpha\hat{\sigma}_y} \;=\; (\cos 2\alpha)\,\hat{\sigma}_x \,+\, (\sin 2\alpha)\,\hat{\sigma}_z \tag{55}$$

with $\alpha = \frac{\pi}{4}$, we arrive at the diagonalized Hamiltonian for eq. (54) as

$$\hat{H}_d \;=\; (n_1+1)\,\hbar\,\omega\,\mathbb{1}_2 \,+\, \tfrac{\hbar\kappa}{\sqrt{2}}\,\sqrt{n_1+1}\,(\sin 2\alpha)\,\hat{\sigma}_z\,. \tag{56}$$

Accordingly, we acquire the energy eigenstates as

$$|\mathbf{1}_2\rangle \;=\; e^{-i\frac{\pi}{4}\hat{\sigma}_y}\,\left|n_1+1,-\tfrac{1}{2}\right\rangle \;=\; \tfrac{1}{\sqrt{2}}\,(|n_1+1,-\tfrac{1}{2}\rangle \,+\, |n_1,\tfrac{1}{2}\rangle)\;,\;\; |\mathbf{2}_2\rangle \;=\; e^{-i\frac{\pi}{4}\hat{\sigma}_y}\,\left|n_1,\tfrac{1}{2}\right\rangle \;=\; \tfrac{1}{\sqrt{2}}\,(-|n_1+1,-\tfrac{1}{2}\rangle \,+\, |n_1,\tfrac{1}{2}\rangle) \tag{57}$$

with the corresponding eigenvalues, $E_1 = (n_1+1)\,\hbar\,\omega - \frac{\hbar\kappa}{\sqrt{2}}\sqrt{n_1+1}$ and $E_2 = (n_1+1)\,\hbar\,\omega + \frac{\hbar\kappa}{\sqrt{2}}\sqrt{n_1+1}$, respectively.

Then, just as in discussion related to eq. (13) for the two coupled linear oscillators, the state evolution for the initial state $|\psi(0)\rangle = \left|n_1, \frac{1}{2}\right\rangle$ is given by

$$|\psi(t)\rangle \;=\; \tfrac{1}{\sqrt{2}}\,|\mathbf{1}_2\rangle\,e^{-\frac{i}{\hbar}E_1 t} - \tfrac{1}{\sqrt{2}}\,|\mathbf{2}_2\rangle\,e^{-\frac{i}{\hbar}E_2 t}\,. \tag{58}$$

This state explicitly includes the possibility of spontaneous emission ($n_1 = 0$). From eq. (58), we easily acquire the reduced density matrix of the angular momentum oscillator as

$$\hat{\rho}^{(2)}(t) \;=\; \mathrm{Tr}_1\,|\psi(t)\rangle\langle\psi(t)| \;=\; \sin^2\left(\tfrac{\kappa t}{\sqrt{2}}\sqrt{n_1+1}\right)\,\left|-\tfrac{1}{2}\right\rangle\left\langle-\tfrac{1}{2}\right| \,+\, \cos^2\left(\tfrac{\kappa t}{\sqrt{2}}\sqrt{n_1+1}\right)\,\left|\tfrac{1}{2}\right\rangle\left\langle\tfrac{1}{2}\right|\,, \tag{59}$$

which represents a mixed state. Therefore, it immediately follows that

$$\mathcal{P}[\hat{\rho}^{(2)}] \;=\; \sin^4\left(\tfrac{\kappa t}{\sqrt{2}}\sqrt{n_1+1}\right) \,+\, \cos^4\left(\tfrac{\kappa t}{\sqrt{2}}\sqrt{n_1+1}\right), \tag{60}$$

with $\mathcal{M}_{|\psi(t)\rangle} = 1 - \mathcal{P}[\hat{\rho}^{(2)}]$ which oscillates between 0 and $\frac{1}{2}$ in the time evolution, thus indicating the appearance of entanglement between the linear and the angular momentum oscillators (see Fig. 3). Furthermore, from eq. (59) we find, after a minor calculation, that

$$\langle\hat{J}_z\rangle \;=\; \mathrm{Tr}_2\,\{\hat{\rho}^{(2)}(t)\cdot\hat{J}_z\} \;=\; \tfrac{1}{2}\;,\; \cos\left(\tfrac{\kappa t}{\sqrt{2}}\sqrt{n_1+1}\right)\;,\langle\hat{J}_z^2\rangle \;=\; \tfrac{1}{4}\;,\;\langle\hat{J}_z^3\rangle \;=\; \tfrac{1}{8}\,\cos\left(\tfrac{\kappa t}{\sqrt{2}}\sqrt{n_1+1}\right)\,. \tag{61}$$

Therefore, from eqs. (46), (61), it follows that

$$\langle\ddot{\hat{J}}_z\rangle \;=\; -2\,(n_1+1)\,\kappa^2\,\langle\hat{J}_z\rangle\,, \tag{62}$$

which obviously yields the consistent result with that in eq. (48) for the case that $n_1 = 0$ and $J_z = j = \frac{1}{2}$.

3.2.2. Case $j=1$

Next, we consider the case of $j=1$. For the total energy $\mathtt{E}=n_1+1$, from eq. (52) with the basis $\{|n_1+2,-1\rangle,\ |n_1+1,0\rangle,\ |n_1,1\rangle\}$, the Hamiltonian takes up the 3×3 irreducible matrix representation

$$\hat{H} \ \doteq \ \begin{pmatrix} \left(n_1+\frac{3}{2}\right)\hbar\omega & \sqrt{n_1+2}\,\hbar\kappa & 0 \\ \sqrt{n_1+2}\,\hbar\kappa & \left(n_1+\frac{3}{2}\right)\hbar\omega & \sqrt{n_1+1}\,\hbar\kappa \\ 0 & \sqrt{n_1+1}\,\hbar\kappa & \left(n_1+\frac{3}{2}\right)\hbar\omega \end{pmatrix}. \tag{63}$$

Based on the generating operators of the group $SU(n)$,[9] where $n=2j+1$ (see Appendix B), the Hamiltonian in eq. (63) can be reexpressed as

$$\hat{H} \ = \ \hbar\,\omega\,\left(n_1+\tfrac{3}{2}\right)\,\mathbb{1}_3 \ + \ \hat{H}_{12}\ ,\ \ \hat{H}_{12} \ = \ \hbar\,\kappa\,\sqrt{n_1+2}\,\hat{u}_{12} \ + \ \hbar\,\kappa\,\sqrt{n_1+1}\,\hat{u}_{23}\,, \tag{64}$$

where $\hat{u}_{12}$ and $\hat{u}_{23}$ are generating operators of $SU(n)$, and their matrix forms are explicitly given in eq. (B3).

Here, it is interesting to note that for a large $n_1\gg 1$ such that $\sqrt{n_1+1}\approx\sqrt{n_1+2}$, the Hamiltonian in eq. (63) can be transformed approximately to

$$\hat{H} \ \simeq \ \left(n_1+\tfrac{3}{2}\right)\,\hbar\,\omega\,\mathbb{1}_3 \ + \ \sqrt{2\,(n_1+2)}\,\hbar\,\kappa\,\hat{J}_x\,, \tag{65}$$

which formally corresponds to the exact Hamiltonian for $j=\frac{1}{2}$ in eq. (54) with $\hat{J}_x=\hat{\sigma}_x/2$. Thus, for a large n_1, we have an approximate diagonalized representation $\hat{H}_d$ for $j=1$, based on the previous result.

For the case of general n_1, proceeding from eq. (63), we easily obtain the energy eigenstates with respect to the basis $\{|n_1+2,-1\rangle,\ |n_1+1,0\rangle,\ |n_1,1\rangle\}$ as

$$|\mathbf{1}_3\rangle \ \doteq \ \frac{1}{\sqrt{2\,(2n_1+3)}}\begin{pmatrix}\sqrt{n_1+2}\\ \sqrt{2n_1+3}\\ \sqrt{n_1+1}\end{pmatrix}\ ,\ |\mathbf{2}_3\rangle \ \doteq \ \frac{-1}{\sqrt{2n_1+3}}\begin{pmatrix}\sqrt{n_1+1}\\ 0\\ -\sqrt{n_1+2}\end{pmatrix}\ ,\ |\mathbf{3}_3\rangle \ \doteq \ \frac{1}{\sqrt{2\,(2n_1+3)}}\begin{pmatrix}\sqrt{n_1+2}\\ -\sqrt{2n_1+3}\\ \sqrt{n_1+1}\end{pmatrix}\,, \tag{66}$$

and the corresponding energy eigenvalues,

$$E_1 \ = \ \left(n_1+\tfrac{3}{2}\right)\,\hbar\,\omega \ + \ \kappa\,\hbar\,\sqrt{2n_1+3}\ ,\ E_2 \ = \ \left(n_1+\tfrac{3}{2}\right)\,\hbar\,\omega\ ,\ E_3 \ = \ \left(n_1+\tfrac{3}{2}\right)\,\hbar\,\omega \ - \ \kappa\,\hbar\,\sqrt{2n_1+3}\,, \tag{67}$$

respectively. Therefore, the diagonalized Hamiltonian is given by

$$\hat{H}_d \ = \ \hbar\,\omega\,\left(n_1+\tfrac{3}{2}\right)\,\mathbb{1}_3 \ + \ \hbar\,\kappa\,\sqrt{2n_1+3}\,\left(\,|\mathbf{1}_3\rangle\,\langle\mathbf{1}_3| \ - \ |\mathbf{3}_3\rangle\,\langle\mathbf{3}_3|\,\right)\,. \tag{68}$$

In order to diagonalize the Hamiltonian $\hat{H}$ in eq. (52) for higher manifolds of j, it is instructive to systematically analyze the above diagonalization of $\hat{H}$ and confirm for $j=\frac{1}{2},1$. In seeking the rotational operation that diagonalizes $\hat{H}$ in eq. (64) to achieve $\hat{H}_d$, it is necessary to consider the unitary, Euler-like transformation

$$\hat{H}_d \ = \ e^{i\alpha_3\,\hat{v}_{13}}\,e^{i\alpha_2\,\hat{v}_{23}}\,e^{i\alpha_1\,\hat{v}_{12}}\,\hat{H}\,e^{-i\alpha_1\,\hat{v}_{12}}\,e^{-i\alpha_2\,\hat{v}_{23}}\,e^{-i\alpha_3\,\hat{v}_{13}} \tag{69}$$

comprising three sequential, non-commuting rotations with respect to $(\hat{v}_{12},\hat{v}_{23},\hat{v}_{13})$ defined in (B4), with Euler angles $(\alpha_1,\alpha_2,\alpha_3)$. Then, as previously accomplished for the case of $j=\frac{1}{2}$ in eqs. (54) - (56), here we similarly choose $(\alpha_1=-\frac{\pi}{2};\cos\alpha_2=\sqrt{n_1+1}/\sqrt{2n_1+3},\ \sin\alpha_2=-\sqrt{n_1+2}/\sqrt{2n_1+3}\,;\ \alpha_3=-\frac{\pi}{4})$ to obtain $\hat{H}_d$ in eq. (68).

This clearly shows that the diagonalization of the Hamiltonian in eq. (64) cannot be described by a simple rotation such as $\hat{R}^{(j=1)}(\alpha,\beta,\gamma)=e^{-i\alpha\hat{J}_z}\,e^{-i\beta\hat{J}_y}\,e^{-i\gamma\hat{J}_z}$ of an angular momentum oscillator,[9] whereas for $j=\frac{1}{2}$, the rotation $\hat{R}^{(j=\frac{1}{2})}(0,\frac{1}{2},0)$ was used in eq. (55) to diagonalize $\hat{H}$ in eq. (54). However, for the case of $j=1$ and large $n_1\gg 1$ given in eq. (65), the diagonalization can be approximately accomplished by $\hat{R}^{(1)}\left(0,\frac{\pi}{2},0\right)$. Finally, it can be shown by induction, for general j, that the diagonalization of the Hamiltonian in eq. (52) is generally characterized by $\frac{1}{2}(n^2-n)$ rotation angles $\{\alpha_1,\alpha_2,\cdots,\alpha_{\frac{1}{2}(n^2-n)}\}$ with respect to the set $\{\hat{v}_{jk}\}$ as given in eq. (B2), where $n=2j+1$.

Let us now consider the state evolution with the initial condition $|\psi(0)\rangle=|n_1,1\rangle$; just as in eq. (13), from eqs. (66), (67) we find that

$$|\psi(t)\rangle \ = \ \sqrt{\tfrac{n_1+1}{2(2n_1+3)}}\ |\mathbf{1}_3\rangle\ e^{-\frac{i}{\hbar}E_1 t} + \sqrt{\tfrac{n_1+2}{2n_1+3}}\ |\mathbf{2}_3\rangle\ e^{-\frac{i}{\hbar}E_2 t} \ + \ \sqrt{\tfrac{n_1+1}{2(2n_1+3)}}\ |\mathbf{3}_3\rangle\ e^{-\frac{i}{\hbar}E_3 t}\,. \tag{70}$$

From this state, we obtain, after some calculations, the reduced density matrix of the angular momentum oscillator,

$$\hat{\rho}^{(2)}(t) \;=\; \mathrm{Tr}_1|\psi(t)\rangle\langle\psi(t)| \;=\; \tfrac{1}{(2n_1+3)^2} \times \Big[4(n_1+1)(n_1+2)\,\sin^4\left(\tfrac{\sqrt{2n_1+3}}{2}\,\kappa t\right)\,|-1\rangle\langle -1| \,+ \\ (n_1+1)(2n_1+3)\,\sin^2\left(\sqrt{2n_1+3}\,\kappa t\right)\;|0\rangle\langle 0| + \left\{2(n_1+1)\,\cos^2\left(\tfrac{\sqrt{2n_1+3}}{2}\,\kappa t\right)\;+\;1\right\}^2\;|1\rangle\langle 1|\Big]\,. \tag{71}$$

This reduced density matrix represents a mixed state with $\mathcal{P}[\hat{\rho}^{(2)}] \le 1$ and $\mathcal{M}_{|\psi(t)\rangle} \ge 0$, directly reflecting the entanglement of the total state $|\psi(t)\rangle$ [here, eq. (71) for $j = 1$ is comparable to eq. (59) for $j = \frac{1}{2}$]. Furthermore, by noting that $|\psi(t+\mathcal{T})\rangle \equiv e^{-i\hat{H}_d\,\mathcal{T}/\hbar}\,|\psi(t)\rangle = |\psi(t)\rangle$ for a given t with the diagonalized form $\hat{H}_d$ in eq. (68), we can easily show that the state evolution in eq. (70) displays periodicity with period $\mathcal{T} = \frac{2\pi}{\sqrt{2n_1+3}\,\kappa}$ as noted in Fig. 3. This means that the time evolution operator $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$, expressed here with respect to the basis $\{\mathbb{1}_3, \hat{\lambda}_p|\, p = 1, 2, \cdots, 3^2-1\}$ given in Appendix B, as

$$\hat{U}(t) \;=\; \mathbb{1}_3 \,+\, \sum_{p=1}^{3^2-1} U_p(t)\,\hat{\lambda}_p\,, \tag{72}$$

where $U_p(t) \equiv \mathrm{Tr}\,\{\hat{U}(t)\,\hat{\lambda}_p\}$ and $U_p(0) = 0$ for all p, has the periodic property that $U_p(t) = U_p(t+\mathcal{T})$.

3.2.3. *Case* $j = \frac{3}{2}$

Let us consider the case of $j = \frac{3}{2}$, where the Hamiltonian of eq. (52) for the total energy $\mathtt{E} = n_1 + \frac{3}{2}$ is given by

$$\hat{H} \;=\; \hbar\,\omega\;(n_1+2)\;\mathbb{1}_4 \,+\, \hat{H}_{12}\;;\;\; \hat{H}_{12} \;=\; \hbar\,\kappa\,\left(\sqrt{\tfrac{3(n_1+3)}{2}}\;\hat{u}_{12} \,+\, \sqrt{2(n_1+2)}\;\hat{u}_{23} \,+\, \sqrt{\tfrac{3(n_1+1)}{2}}\,\hat{u}_{34}\right)\,. \tag{73}$$

The diagonalized form of eq. (73) can be obtained, after tedious calculations similar to those described in the previous case, as

$$\hat{H}_d \;=\; \hbar\,\omega\;(n_1+2)\;\mathbb{1}_4 \,+\, \tfrac{\kappa\hbar}{2}\,\left\{\,\sqrt{\theta_1+\theta_2}\;(\,|\mathbf{1}_4\rangle\,\langle\mathbf{1}_4| \,-\, |\mathbf{2}_4\rangle\,\langle\mathbf{2}_4|\,) \,+\, \sqrt{\theta_1-\theta_2}\;(\,|\mathbf{3}_4\rangle\,\langle\mathbf{3}_4| \,-\, |\mathbf{4}_4\rangle\,\langle\mathbf{4}_4|\,)\,\right\}\,, \tag{74}$$

where $\theta_1 = 20 + 10\,n_1$; $\theta_2 = 2\sqrt{73 + 64n_1 + 16n_1^2}$, and the $|\mathbf{p}_4\rangle$'s with $\mathbf{p} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ denote energy eigenvectors.

Due to the complex form of the reduced density matrix of the angular momentum oscillator, $\hat{\rho}^{(2)}(t) = \mathrm{Tr}_1|\psi(t)\rangle\langle\psi(t)|$, and the measure of entanglement, $\mathcal{M}_{|\psi(t)\rangle} = 1 - \mathcal{P}[\hat{\rho}^{(2)}(t)]$, obtained from $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle$ with $|\psi(0)\rangle = |n_1, \frac{3}{2}\rangle$, we simply plot the exact numerical results of $\mathcal{M}_{|\psi(t)\rangle}$ in Figs. 4, 5.

Based on the numerical analyses[17] and the statistical approximation in terms of classical random variables[3], it has been shown that $\langle\hat{J}_z(t)\rangle$ with $|\psi(0)\rangle = |0, j\rangle$ exhibits aperiodic behavior in its time evolution for all $j \ge \frac{3}{2}$, even for j large enough (i.e., in the classical limit for an uncoupled angular momentum oscillator), while the classical counterpart displays periodic motion on the spherical pendulum [cf. eq. (51) with $\lambda_1 = \lambda_2 = 0$ and $K = 0$; note that for this initial state, the angular momentum K about the vertical axis through the center vanishes]. This shows that the aperiodicity in the time evolution would be of non-classical origin. We indicate this aperiodicity for any $n = 2j+1 \ge 4$ here by noting the impossibility of having the periodic property $\hat{U}(t) \equiv e^{-i\hat{H}t/\hbar} = \mathbb{1}_n$ for arbitrary t; from eq. (74), we find, after some calculations, that

$$\hat{U}(t) \;=\; e^{-\frac{i}{\hbar}\,\hat{H}_d\,t} \\ \;=\; e^{-i\omega t(n_1+2)}\,(\,e^{-i\frac{\kappa}{2}\,\sqrt{\theta_1+\theta_2}\,t}\,|\mathbf{1}_4\rangle\langle\mathbf{1}_4| \,+\, e^{i\frac{\kappa}{2}\,\sqrt{\theta_1+\theta_2}\,t/2}\,|\mathbf{2}_4\rangle\langle\mathbf{2}_4| \,+\, e^{-i\frac{\kappa}{2}\,\sqrt{\theta_1-\theta_2}\,t}\,|\mathbf{3}_4\rangle\langle\mathbf{3}_4| \,+\, e^{i\frac{\kappa}{2}\,\sqrt{\theta_1-\theta_2}\,t}\,|\mathbf{4}_4\rangle\langle\mathbf{4}_4|\,)\,. \tag{75}$$

Since here the phase factors, $\kappa\,\sqrt{\theta_1+\theta_2}/2$ and $\kappa\,\sqrt{\theta_1-\theta_2}/2$, are incommensurable with respect to each other, we immediately note that not all $U_k(t)$ in the expression

$$\hat{U}(t) \;=\; \mathbb{1}_4 \,+\, \sum_{k=1}^{4^2-1} U_k(t)\,\hat{\lambda}_k \tag{76}$$

can simultaneously vanish periodically, which leads to the aperiodicity in the state evolution $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$ (see Figs. 4, 5). Along the same line, it can be shown that we have the diagonalized Hamiltonian for any j in form

$$\hat{H}_d \;=\; \hbar\,\omega\,\left(n_1+\tfrac{1}{2}+j\right)\,\mathbb{1}_n \,+\, \kappa\,\hbar\,\{\beta_1\;(\,|\mathbf{1}_n\rangle\,\langle\mathbf{1}_n| \,-\, |\mathbf{2}_n\rangle\,\langle\mathbf{2}_n|\,) \,+\, \cdots \,+ \\ \beta_m\;(\,|(\mathbf{2m-1})_n\rangle\,\langle(\mathbf{2m-1})_n| \,-\, |(\mathbf{2m})_n\rangle\,\langle(\mathbf{2m})_n|\,) \,+\, \cdots + \beta_{n/2}\;(\,|(\mathbf{n-1})_n\rangle\,\langle(\mathbf{n-1})_n| \,-\, |\mathbf{n}_n\rangle\,\langle\mathbf{n}_n|\,)\,\}\;, \tag{77}$$

where $n = 2j+1$, and $\beta_m = \beta_m(n_1)$ with $m = 1, 2, \cdots, \frac{n}{2}$; $|\mathbf{p}_n\rangle$'s denote energy eigenvectors. Here, each β_{m_1} is, in general, incommensurable with any other β_{m_2}, where $m_1 \neq m_2$, and for the case that n is odd, one of β_m's always vanishes. Therefore, we have

$$\begin{aligned} \hat{U}(t) &= e^{-\frac{i}{\hbar}\hat{H}_d\, t} = e^{-i\omega t(n_1+\frac{1}{2}+j)} \left(e^{-i\kappa\,\beta_1\, t}\, |\mathbf{1}_n\rangle\langle\mathbf{1}_n| + e^{i\kappa\,\beta_1\, t}\, |\mathbf{2}_n\rangle\langle\mathbf{2}_n| + \cdots + e^{-i\kappa\,\beta_m\, t}\, |(\mathbf{2m-1})_n\rangle\langle(\mathbf{2m-1})_n| + \right. \\ & \left. e^{i\kappa\,\beta_m\, t}\, |(\mathbf{2m})_n\rangle\langle(\mathbf{2m})_n| + \cdots + e^{-i\kappa\,\beta_{n/2}\, t}\, |(\mathbf{n-1})_n\rangle\langle(\mathbf{n-1})_n| + e^{i\kappa\,\beta_{n/2}\, t}\, |\mathbf{n}_n\rangle\langle\mathbf{n}_n| \right), \end{aligned} \tag{78}$$

thus showing that $\hat{U}(t) \neq \mathbb{1}_n$ for any t unless all β_m are commensurable with each other. From this, we easily find that the aperiodic behavior in the state evolution, $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$, will survive and even increase with $n \geq 4$, i.e., $j \geq \frac{3}{2}$ [note that for $j = 1$, we obviously have β_1 only, which leads to the periodicity in the state evolution: $\hat{U}(t) = \mathbb{1}_3$ for the case that $\kappa\,\beta_1\, t = \pm 2\pi, \pm 4\pi, \pm 6\pi, \cdots$].

4. CONCLUSIONS

In summary, we have investigated the fundamental dynamics of two interacting oscillators: in one scenario, two linear oscillators, and in the other scenario, a linear and a non-linear oscillator have been considered. For the first scenario, based on the coupled boson representation we compactly described the equation of motion in the Heisenberg picture and also systematically studied the quantum entanglement in the time evolution of the total wave-function $|\psi(t)\rangle$ for various initial states in the Schrödinger picture. From this, we found that the appearance of entanglement in the time evolution from the disentangled initial simple product state is attributed to the interaction between the two individual oscillators. Also, the measure of entanglement increases with the total occupation number.

For the second scenario, quantum versus classical behaviors have been studied, based on the Heisenberg equation developed in terms of relevant reduced kinematics operator variables and parameterized commutator relations. By setting the corresponding commutator relations to one or zero, respectively, the Heisenberg equations are shown to describe the full quantum or classical motion of the interaction system, thus allowing us to discern the differences between the fully quantum and fully classical dynamical picture. In addition, for this second scenario, in the fully quantum-mechanical description, we considered special examples of $j = \frac{1}{2}, 1, \frac{3}{2}$ for the coupled angular momentum state, demonstrating the explicit appearances of entanglement. The entanglement increases with j and so persists even as $j \to \infty$, a limit which would go over to the classical picture for an *uncoupled* angular momentum oscillator. This entanglement occurs because the uppermost excited state of the j-level angular momentum oscillator is always *coupled* to the vacuum state of the linear oscillator, a purely quantum coupling which manifestly gives rise to spontaneous emission. We have also shown that the dynamics of this scenario can be systematically described by the irreducible matrix representation based on the generating operators of the group $SU(n)$, while that of the first scenario can be given, based on the coupled boson representation, simply by the rotations of an angular momentum oscillator. For the coupled linear-angular momentum oscillator, this system was shown to display periodicity in the measure of entanglement for $j = \frac{1}{2}$ and $j = 1$, whereas for $j = \frac{3}{2}$ and beyond, the measure of entanglement was shown to be aperiodic; this aperiodicity is apparent from the form of the diagonalized multi-level Hamiltonian and the resulting structure of the time evolution operator.

5. ACKNOWLEDGMENTS

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APPENDIX A: MATHEMATICAL SUPPLEMENTS TO EQ. (23)

From eqs. (21) and (23) with the relation that $U^{(J)}_{m\,\mathbf{k}}(\gamma) = U^{(J)}_{\mathbf{k}\,m}(-\gamma)$, we find that

$$\begin{aligned} f^{(1)}_{\mathbf{k}}(t) &= (-1)^M \sqrt{\tfrac{(J+M)!\,(J-M)!}{(J+\mathbf{k})!\,(J-\mathbf{k})!}}\, \left(\sin\tfrac{\gamma}{2}\right)^{M-\mathbf{k}} \left(\cos\tfrac{\gamma}{2}\right)^{\mathbf{M+k}} \times \\ & \sum_{m=-J}^{J} (-1)^m \left(\cos\tfrac{\gamma}{2}\right)^{2m} \cdot \left\{\mathcal{P}^{(M-m,M+m)}_{J-M}(\cos\gamma)\right\} \cdot \left\{\mathcal{P}^{(m-\mathbf{k},m+\mathbf{k})}_{J-m}(\cos\gamma)\right\} \cdot e^{-im\bar{\omega}t}\,. \end{aligned} \tag{A1}$$

Particularly for the case that $\gamma = \frac{\pi}{2}$, namely $\omega_1 = \omega_2$, eq. (A1) reduces to

$$f_{\mathbf{k}}^{(1)}(t) \;=\; \frac{1}{4^J} \sqrt{\frac{(J+M)!\,(J-M)!}{(J+\mathbf{k})!\,(J-\mathbf{k})!}} \sum_{m=-J}^{J} e^{-im\bar{\omega}t} \cdot \sum_{p=0}^{J-M} \sum_{q=0}^{J-m} (-1)^{p+q} \cdot \binom{J-m}{p} \binom{J+m}{J-M-p} \binom{J-\mathbf{k}}{q} \binom{J+\mathbf{k}}{J-m-q}\,, \tag{A2}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.

APPENDIX B: GENERATING OPERATORS OF THE GROUP SU(N)

The set of the generating operators is given by[9]

$$\hat{\boldsymbol{\lambda}} \;=\; \{\hat{\lambda}_p |\, p = 1, 2, \cdots, n^2-1\} \;=\; \{\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23}, \cdots, \hat{v}_{12}, \hat{v}_{13}, \hat{v}_{23}, \cdots, \hat{w}_1, \hat{w}_2, \cdots, \hat{w}_{n-1}\}\,, \tag{B1}$$

where $|\hat{\boldsymbol{\lambda}}| = n^2 - 1$, and

$$\hat{u}_{kl} \;=\; \hat{P}_{kl} + \hat{P}_{lk}\;;\;\; \hat{v}_{kl} \;=\; i\,(\hat{P}_{kl} - \hat{P}_{lk})\;;\;\; \hat{w}_m \;=\; -\sqrt{\frac{2}{m(m+1)}}\,(\hat{P}_{11} + \hat{P}_{22} + \cdots + \hat{P}_{mm} - m\hat{P}_{m+1,m+1})\,. \tag{B2}$$

Here, $1 \le k < l \le n$, and $|\{\hat{u}_{kl}\}| = |\{\hat{v}_{kl}\}| = \frac{1}{2}(n^2 - n)$, where $\{\hat{u}_{kl}\}$ and $\{\hat{v}_{kl}\}$ denote $\{\hat{u}_{12}, \hat{u}_{13}, \hat{u}_{23}, \cdots, \hat{u}_{n-1,n}\}$ and $\{\hat{v}_{12}, \hat{v}_{13}, \hat{v}_{23}, \cdots, \hat{v}_{n-1,n}\}$, respectively; $1 \le m \le n-1$, and $|\{\hat{w}_m\}| = n-1$, where $\{\hat{w}_m\}$ denotes $\{\hat{w}_1, \hat{w}_2, \cdots, \hat{w}_{n-1}\}$. $\hat{P}_{kl} = |k\rangle\langle l|$ represents a projection operator for $k = l$, and a transition operator for $k \ne l$. For $n = 2$ (i.e., $j = \frac{1}{2}$), the generators $(\hat{u}_{12}, \hat{v}_{12}, \hat{w}_1)$ exactly correspond to $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. For $n = 3$ (i.e., $j = 1$), three generators $\{\hat{u}_{12}, \hat{u}_{23}, \hat{u}_{13}\}$, for example, have the matrix representations,

$$\hat{u}_{12} \;=\; \hat{P}_{12} \,+\, \hat{P}_{21} \;\doteq\; \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,\; \hat{u}_{23} \;=\; \hat{P}_{23} \,+\, \hat{P}_{32} \;\doteq\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,,\; \hat{u}_{13} \;=\; \hat{P}_{13} \,+\, \hat{P}_{31} \;\doteq\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \tag{B3}$$

respectively. Similarly, for 3 generators $\{\hat{v}_{12}, \hat{v}_{23}, \hat{v}_{13}\}$, we have

$$\hat{v}_{12} \doteq \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,\; \hat{v}_{23} \doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}\,,\; \hat{v}_{13} \doteq \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\,. \tag{B4}$$

The set $\{\mathbb{1}_n, \hat{\lambda}_p |\, p = 1, 2, \cdots, n^2-1\}$ can be used as a well-defined basis of the space of $n \times n$ matrices.

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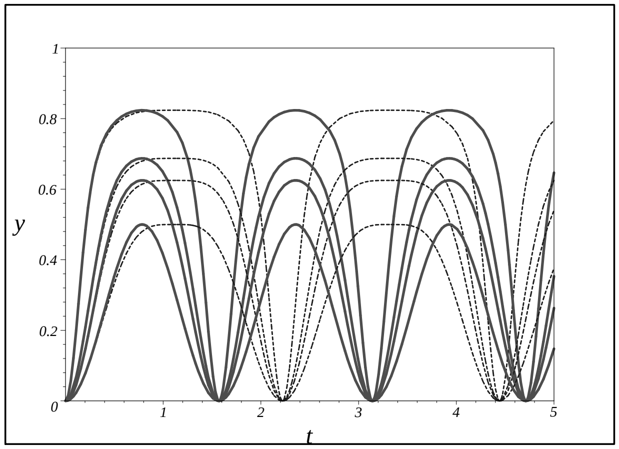

FIG. 1: The measure of entanglement, $y(t) = \mathcal{M}_{|\psi(t)\rangle}$ given in eq. (24) versus time t is depicted for $\kappa = 1$ and initial states $|\psi(0)\rangle = |N\rangle|0\rangle = |J; J\rangle$, where $N = 1, 2, 3, 10$ (i.e., $J = \frac{1}{2}, 1, \frac{3}{2}, 5$) in progressive order from bottom to top; the solid lines are obtained for $\gamma = \frac{\pi}{2}$ (or $\omega_1 = \omega_2$), while the dash lines for $\gamma = \frac{\pi}{4}$ (or $\omega_1 - \omega_2 = 2$).

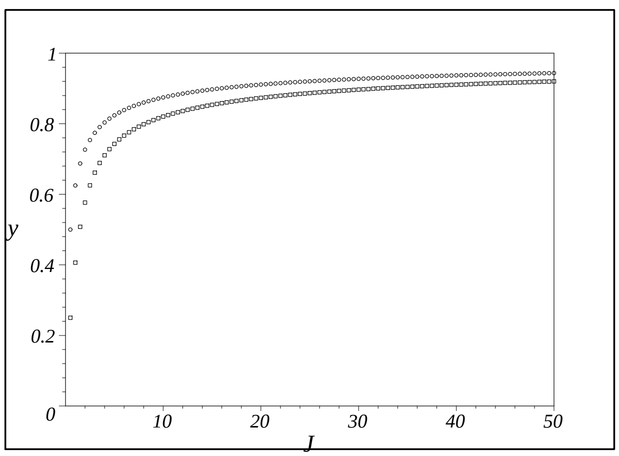

FIG. 2: The measure of entanglement, $y = \mathcal{M}_{|\psi(t)\rangle}$, as given in eq. (28), versus J is depicted for $\kappa = 1$ and initial states $|\psi(0)\rangle = |J; J\rangle_H = |N, 0\rangle_H$ [cf. eq. (29)]; the circles ($\circ$) are used to indicate $\gamma = \frac{\pi}{2}$ while the boxes ($\square$) indicate $\gamma = \frac{\pi}{4}$. As noted in eq. (27), y is time-independent.

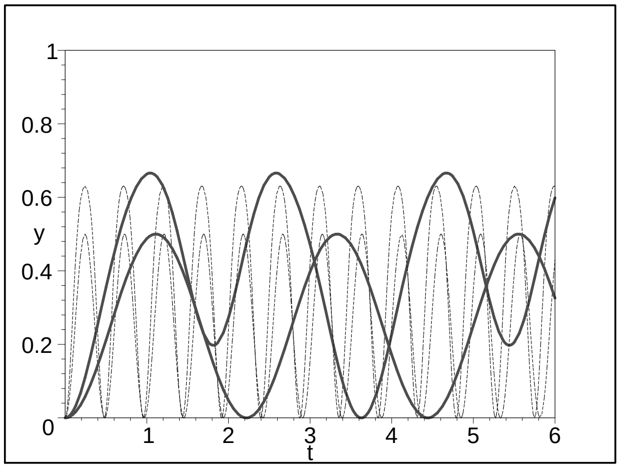

FIG. 3: The measure of entanglement, $y(t) = \mathcal{M}_{|\psi(t)\rangle}$ versus time t is depicted for $\kappa = 1$, $\Delta\omega = 0$, and initial states $|\psi(0)\rangle = |n_1, j\rangle$; the solid lines are obtained for $n_1 = 0$, while the dash lines for $n_1 = 20$; $j = \frac{1}{2}, 1$ from bottom to top [cf. eqs. (60), (71)]; for $j = \frac{1}{2}$, the current $y(t)$ takes up the form of the $y(t)$ given in Fig. 1; for $j = 1$, the current $y(t)$ becomes closer in form to the $y(t)$ given in Fig. 1 with the increase of n_1.

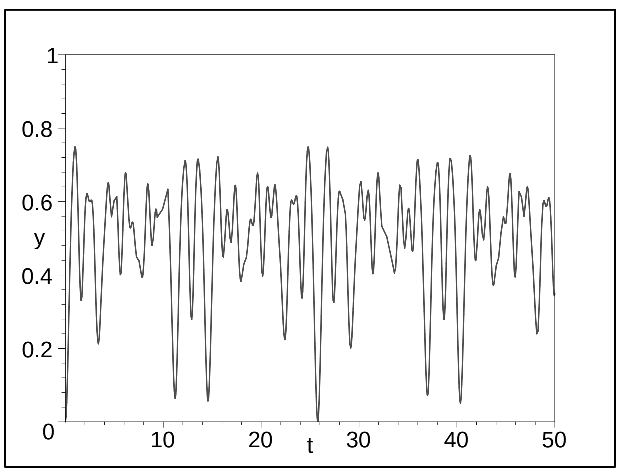

FIG. 4: The measure of entanglement, $y(t) = \mathcal{M}_{|\psi(t)\rangle}$ versus time t is depicted for $\kappa = 1$, $\Delta\omega = 0$, and $|\psi(0)\rangle = |n_1 = 0,\, j = \frac{3}{2}\rangle$; we see here the the aperiodic behavior of $y(t)$, and its maximum $y_{\mathbf{m}}$ is larger than maxima of the $y(t)$ (solid lines) for $j = \frac{1}{2}, 1$ given in Fig. 3.

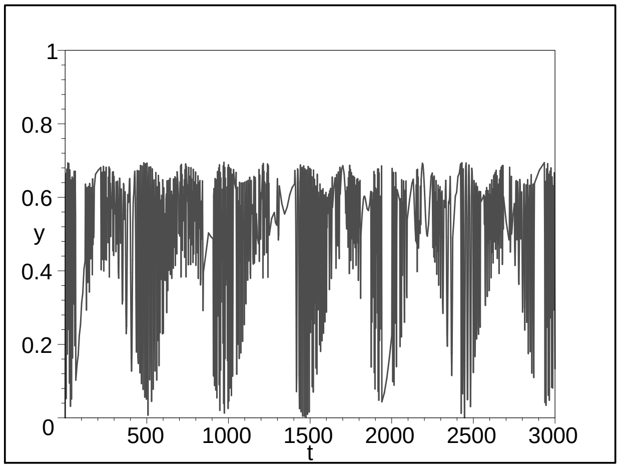

FIG. 5: The measure of entanglement, $y(t) = \mathcal{M}_{|\psi(t)\rangle}$ versus time t is depicted for $\kappa = 1$, $\Delta\omega = 0$, and $|\psi(0)\rangle = |n_1 = 20,\, j = \frac{3}{2}\rangle$; we also here see the the aperiodic behavior of $y(t)$, and its maximum $y_{\mathbf{m}}$ is larger than maxima of the $y(t)$ (dash lines) for $j = \frac{1}{2}, 1$ given in Fig. 3.